Gabriel's Horn

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Abstract

Can an object have a constant and finite volume yet an infinite and incalculable surface area?

An object with these properties is impossible to represent in the real world. However, in theory, mathematics may be able to express this.

Introduction

Evangelista Torricelli was an Italian physicist and mathematician, and a student of Galileo.

Torricelli was the first person to begin exploring the idea and studying the possibility of an object with the properties of Gabriel's horn.

If the function f(x) = 1/x is expressed after revolving about the x-axis from x = 1 to x = ∞, the resulting solid of revolution is Gabriel's horn. This is the shape that will be used for reference and can be seen in Figure 1.

Mathematics

Using calculus, a better understanding of this shape's volume and surface area can be achieved.

Volume (Disk method)

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx$$

$$\lim_{t\to\infty}\left(\pi \int_1^t \left(\frac{1}{x}\right)^2 dx\right)$$

$$\pi \cdot \lim_{t\to\infty}\left(1 - \frac{1}{t}\right)$$

$$\pi$$

Surface Area

$$SA = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx$$

Since

$$\lim_{t\to\infty}\left(2\pi \int_1^t \frac{1}{x} dx\right) = 2\pi \cdot \lim_{t\to\infty}\left(\ln(t) - \ln(1)\right)$$

and

$$2\pi \cdot \lim_{t\to\infty}\left(\ln(t) - \ln(1)\right) = \infty$$

This means

$$SA > 2\pi \int_1^{\infty} \frac{1}{x}\, dx = \infty$$

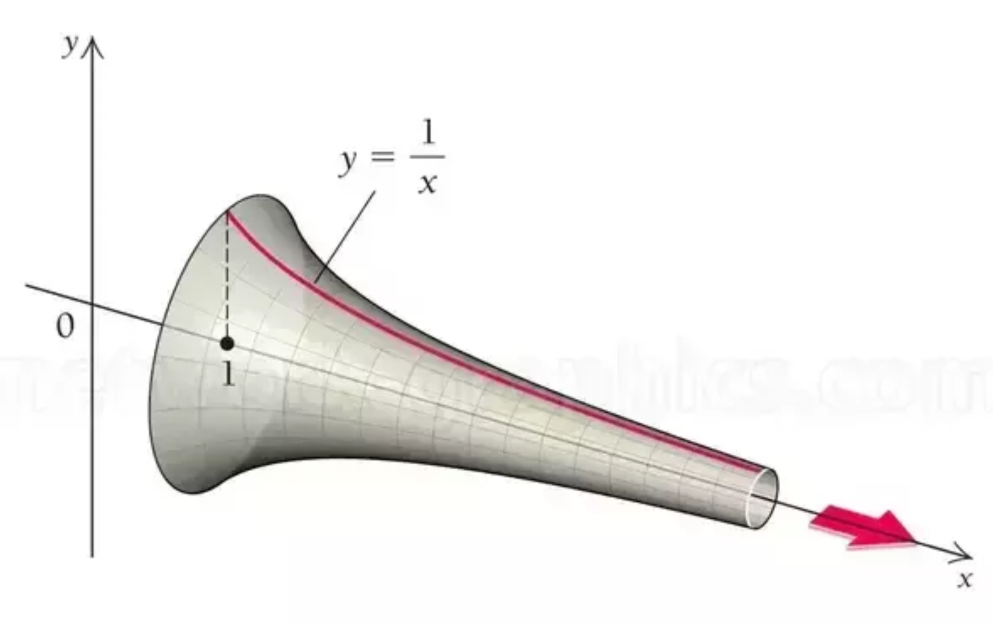

Figure 1: A visual representation of the solid of revolution described.

Figure 2: A portrait of Evangelista Torricelli

Summary

For the volume calculation, the function's value is squared (the radius), multiplied by π, and integrated over the mentioned interval (1, ∞) (disk thickness). After calculation and simplification it is apparent that this object's volume is finite and constant. π to be exact.

However, when attempting to calculate the surface area the answer is not so evident. Calculating surface area requires a multiplication of the circumference (length around the solid at any instant) by the arc length (length of the solid). After first comparing this integral to the calculation of one simpler and smaller (considering the term under the radical will always be ≥ 1) and discovering it's divergence, it is concluded that the initial and larger equation must also diverge to infinity.

It is now clear that there can indeed be an object (expressed with mathematics) that has a finite volume yet infinite surface area.

References

"Gabriel's Horn." Brilliant, brilliant.org/wiki/gabriels-horn/.

"Gabriel's Horn Is Not so Puzzling." The Universe of Discourse, blog.plover.com/math/gabriels-horn.html.

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